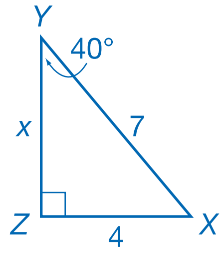
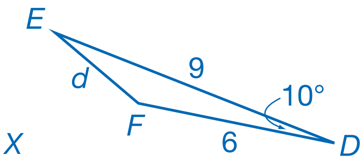
LESSON

15 Trigonometric Laws

Law of Sines

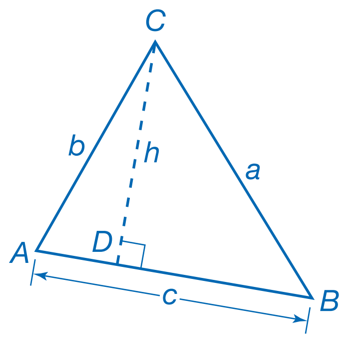
UNDERSTAND Triangle *XYZ* is a right triangle, so m$\angle X$ is the compliment of 40° and the length of side $\overline{YX}$ can be found using the cosine ratio of 40°. Triangle *DEF* is not a right triangle, so it has no hypotenuse or right angle. The measures of $\angle E$ and $\angle F$ and the length of side $\overline{EF}$ cannot be found so easily.





In earlier lessons, you learned how to use trigonometric functions to find missing side lengths and angle measures in right triangles. Now, you will see that trigonometric functions can help you find missing measurements in any triangle, as long as at least three other measurements are known. To find unknown angles and sides in nonright triangles, you must apply the trigonometric laws.

Altitude $\overline{CD}$ with length h divides $\triangle ABC$ on the right into two right triangles. Sine ratios for angles A and B can be written using these triangles. Solve both equations for h.



$\sin B = \frac{h}{a}$ $\qquad$ $\sin A = \frac{h}{b}$

$a \sin B = h$ $\qquad$ $b \sin A = h$

Use the transitive property of equality to combine the two equations into one.

$a \sin B = b \sin A$ $\qquad$ Divide both sides by $(\sin A)(\sin B)$.

$\frac{a}{\sin A} = \frac{b}{\sin B}$

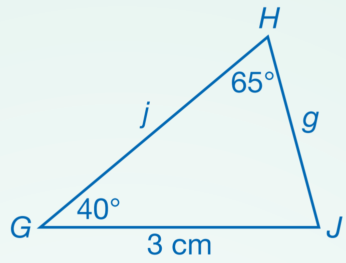
By repeating this process for another altitude of $\triangle ABC$, it can be shown that these ratios are also equivalent to $\frac{c}{\sin C}$.

Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, where a, b, and c are the lengths of the sides of a triangle and A, B, and C are their opposite angles.

The Law of Sines can be used when one angle in a triangle and the length of the side opposite that angle are known. As long as one other side length or angle measure is known, all other measurements can be found.

Connect

Find g and j.



1 Find the unknown angle measure.

The angle measures of a triangle sum to 180°

$\angle G + \angle H + \angle J = 180°$

$40° + 65° + \angle J = 180°$

$\angle J = 75°$

2 Write a proportion using the Law of Sines.

Write a general rule and then substitute in the known values.

$\frac{g}{\sin G} = \frac{h}{\sin H} = \frac{j}{\sin J}$

$\frac{g}{\sin 40°} = \frac{3}{\sin 65°} = \frac{j}{\sin 75°}$

3 Solve for g.

$\frac{g}{\sin 40°} = \frac{3}{\sin 65°}$

$g = \frac{3}{\sin 65°} \cdot \sin 40°$

▶ $g \approx 2.13$ cm

4 Solve for j

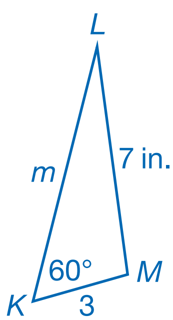
$\frac{j}{\sin 75°} = \frac{3}{\sin 65°}$

$j = \frac{3}{\sin 65°} \cdot \sin 75°$

▶ $j \approx 3.20$ cm

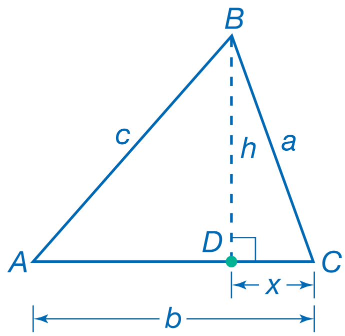
TRY

Find m$\angle L$, m$\angle M$, and length of side m.



Law of Cosines

UNDERSTAND Altitude $\overline{BD}$ with length h divides $\triangle ABC$ below into two right triangles.



By using right triangle CBD, we can write the following trigonometric ratios:

$\sin C = \frac{h}{a}$ $\qquad \cos C = \frac{x}{a}$

$a\sin C = h$ $\qquad a\cos C = x$

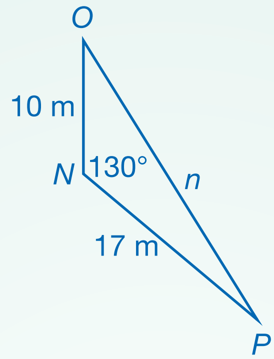
Apply the Pythagorean Theorem to $\triangle ABD$, which has a hypotenuse of length c and legs of length h and $b - x$. Substitute expressions for h and x from the equations above.

$c^2 = h^2 + (b - x)^2$	Use the Pythagorean Theorem.
$c^2 = (a\sin C)^2 + (b - x)^2$	Substitute $a\sin C$ for h.
$c^2 = (a\sin C)^2 + (b - a\cos C)^2$	Substitute $a\cos C$ for x.
$c^2 = a^2 \sin^2 C + b^2 - 2ab\cos C + a^2\cos^2 C$	Expand the binomial.
$c^2 = a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab\cos C$	Factor out a^2.
$c^2 = a^2 + b^2 - 2ab\cos C$	Substitute 1 for $\sin^2 C + \cos^2 C$.

Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$, where C denotes the angle of a triangle contained between sides of lengths a and b and opposite the side of length c.

Connect

Find the length of side n, $m\angle O$, and $m\angle P$ in $\triangle NOP$.



1 Use the Law of Cosines to find n.

Apply the Law of Cosines, letting $a = 10$, $b = 17$, $c = n$, and $C = 130°$.

$c^2 = a^2 + b^2 - 2ab\cos C$

$n^2 = (10)^2 + (17)^2 - 2(10)(17)\cos(130°)$

$n^2 = 100 + 289 - 340\cos 130°$

$n^2 \approx 100 + 289 - 340(-0.6428)$

$n^2 \approx 100 + 289 + 218.5$

$n^2 \approx 607.5$

$n \approx 24.6$

▶ The missing side length is approximately 24.6 meters.

2 Use the Law of Cosines to find the measure of one angle.

Let $a = 10$, $b = 24.6$, $c = 17$, and $C = m\angle O$.

$c^2 = a^2 + b^2 - 2ab\cos C$

$17^2 = (10)^2 + (24.6)^2 - 2(10)(24.6)\cos O$

$289 = 100 + 605.16 - 492\cos(O)$

$289 = 705.16 - 492\cos O$

$0 = 416.16 - 492\cos O$

Isolate $\cos O$ and use the inverse cosine function to find $m\angle O$.

$492\cos O = 416.16$

$\cos O \approx 0.8459$

$\cos^{-1}(\cos O) \approx \cos^{-1}(0.8459)$

▶ $m\angle O \approx 32°$

3 Find the measure of the remaining angle.

The angle measures of a triangle sum to 180°.

$m\angle N + m\angle O + m\angle P = 180°$

$130° + 32° + m\angle P \approx 180°$

▶ $m\angle P \approx 18°$

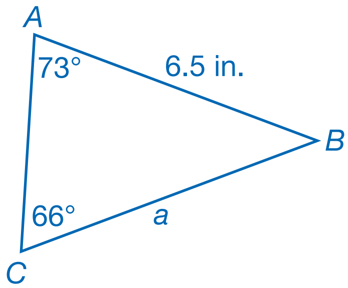
TRY

A triangle has side lengths 4, 8, and 10. Find the approximate angles of the triangle.

Practice

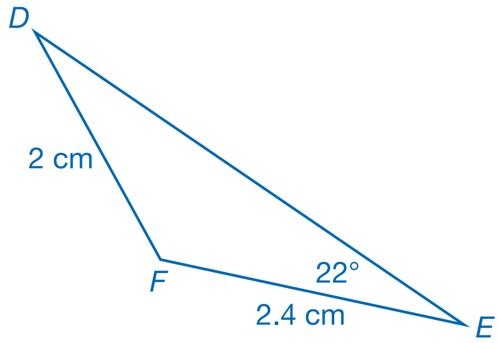
Use the Law of Sines to find the missing measurements. Round to the nearest tenth.

1.



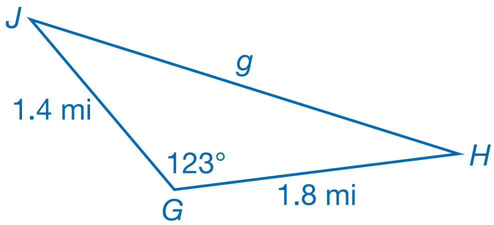
$a =$ ____________

2.



$m\angle D =$ ____________

Use $\triangle GHJ$ below for questions 3 and 4. Choose the best answer.



3. What is the length of g?

A. 1.2 mi

B. 2.8 mi

C. 5.1 mi

D. 7.9 mi

HINT In the Law of Cosines, C is the angle between sides a and b.

4. What is $m\angle H$?

A. 8.5°

B. 12.2°

C. 24.8°

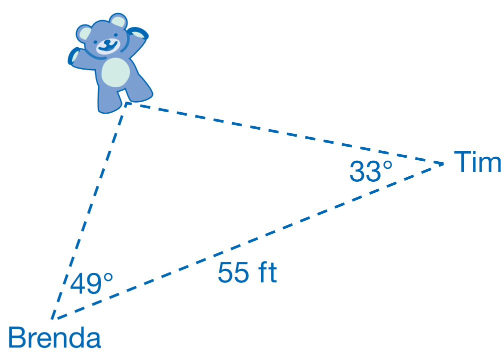
D. 32.6°

Solve.

5. Brenda and Tim are on a scavenger hunt. They are 55 feet apart when they both spot the last item on their list, a blue teddy bear. The diagram to the right shows their positions.

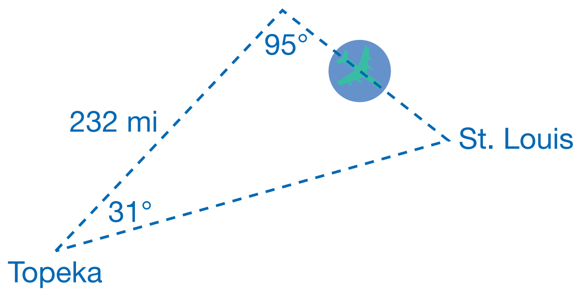
Who is closer to the teddy bear? By approximately how much?

____



Solve.

6. A plane is flying from Topeka to St. Louis. In order to avoid a tornado, the pilot diverts the plane 31° from the original flight path. After flying 232 miles, the plane turns at an angle of 95° back toward St Louis.

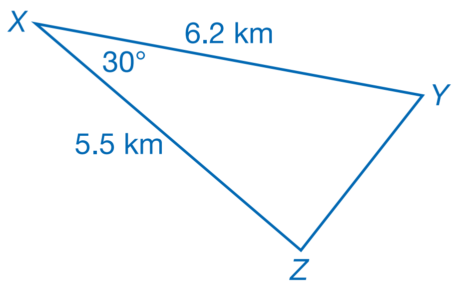


What is the total length of the modified flight path, to the nearest mile? ____________________

About how much farther did the plane travel than it would have if it had flown a direct route?

____

7. **APPLY** A surveyor took some measurements of a triangular plot of land that is for sale. He made the following sketch of his measurements.



What is the approximate perimeter of the land? ____________________

What is the approximate area of the land? ____________________

If the plot of land is sold for $29,000, what is the approximate price per square kilometer?

____

8. **GENERALIZE** When does it make sense to use the Law of Sines? When does it make sense to use the Law of Cosines?

____

____

____

____